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Chern–Simons gauge theory on orbifolds: Open strings from three dimensions

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Abstract

Chern–Simons gauge theory is formulated on three-dimensional $\mathbb{Z}_2$ orbifolds. The locus of singular points on a given orbifold is equivalent to a link of Wilson lines. This allows one to reduce any correlation function on orbifolds to a sum of more complicated correlation functions in the simpler theory on manifolds. Chern–Simons theory on manifolds is known to be related to two-dimensional (2D) conformal field theory (CFT) on closed-string surfaces; here it is shown that the theory on orbifolds is related to 2D CFT of unoriented closed- and open-string models, i.e. to worldsheet orbifold models. In particular, the boundary components of the worldsheet correspond to the components of the singular locus in the 3D orbifold. This correspondence leads to a simple identification of the open-string spectra, including their Chan–Paton degeneration, in terms of fusing Wilson lines in the corresponding Chern–Simons theory. The correspondence is studied in detail, and some exactly solvable examples are presented. Some of these examples indicate that it is natural to think of the orbifold group $\mathbb{Z}_2$ as a part of the gauge group of the Chern–Simons theory, thus generalizing the standard definition of gauge theories.

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1. Introduction

Since the first appearance of the notion of "orbifolds" in Thurston's 1977 lectures on three-dimensional topology [1], orbifolds have become very appealing objects for physicists. This

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interest was mainly motivated by the fact that orbifold singularities are so mild that strings can propagate consistently on orbifold targets, without violating unitarity of the string S matrix [2].

In critical string theory, orbifolds as targets for string propagation have been generalized to more subtle structures. One example is asymmetric orbifolds [3], i.e. two-dimensional (2D) conformal field theories (CFTs) in which left-movers and right-movers take values in distinct orbifolds. The geometrical structure of these generalizations becomes more involved: Intriguing subtleties come into play, related in particular to the geometry of fixed points and the vacuum degeneracy of twisted sectors [3]. Another generalization is given by worldsheet orbifolds [4] (see also [5,6] for related points of view), where string theoretic vacua are orbifolded by a symmetry that acts directly on the 2D CFTs describing the vacua. It is within this construction that open strings emerge as twisted states of an orbifold. Since the orbifold interpretation of open-string models first arose, the technique has been useful in several instances where one is interested in the open-string counterparts of closed-string constructions (such as target duality [7] or 2D black holes [8]).

While now we understand fairly well that open strings come from twisted sectors in a class of generalized orbifold models, the geometry of this generalization is not yet completely understood. Indeed, the target orbifold geometry gets here even more entangled with the structure of the conformal field theory itself. On the other hand, one not fully understood issue in open-string theory is the degeneration of the ground state in the open-string sector of a given model. Traditionally, this degeneration is constructed by the Chan–Paton mechanism, which eventually leads to the presence of non-abelian Yang–Mills gauge symmetry in the space–time theory. In the Chan–Paton mechanism, the degeneration is caused by the somewhat ad hoc procedure of inserting charges of a space–time gauge group (typically SO(N)) at the ends of open strings. The seeming arbitrariness in the choice of the gauge group is eventually fixed by the check of BRST invariance of the string model, which leads to an essentially unique gauge symmetry group for each model. This state of affairs seems unsatisfactory, and a deeper explanation of the existence and of the high degree of uniqueness of the Chan–Paton mechanism is to be sought. In fact, some hints are offered by the orbifold construction of open-string theory: Open strings belong to twisted sectors on orbifolds, and one may expect connections between their vacuum degeneracy and some sort of generalized fixed-point geometry of the orbifold [4,7]. I will clarify some of these subtleties in this paper, making use of a higher-dimensional perspective.

Let us now leave the stringy intricacies aside and consider something simpler, namely a quantum field theory on orbifolds. Doing this, however, one typically encounters inconsistencies: the scattering S matrix of local excitations is not unitary, as particles may leave the world through the singular points. There is, however, one important loophole in this argument. Were we considering a topological quantum field theory, there would be no local excitations, no S matrix, and hence no violation of the S matrix unitarity. Thus, we are free to construct a quantum field theory on orbifolds, on condition that the theory has no local excitations, i.e. that it is topological.

In this paper, we will be concerned with Chern–Simons (CS) gauge theory [9] on 3D orbifolds. One motivation for this is string theoretical. Below I will argue that CS gauge theory on 3D $\mathbb{Z}_2$ orbifolds is related to the theory of open strings in precisely the same sense as CS gauge theory on manifolds is related to the theory of closed strings (or more precisely, to rational CFT on compact oriented surfaces of closed-string theory). This will give us the higher-dimensional perspective of the puzzles of open-string theory that I mentioned above. The 3D vantage point as a tool explaining various properties of 2D CFT has been advocated by Witten [10] in the context of CFTs on closed oriented Riemann surfaces; it is the open-string extension of this ideology that is new in this paper. Another motivation for the present work may come from the fact that the CS gauge theory on orbifolds represents an explicit example of equivariant topological quantum field theory in the sense of the axiomatics presented in [11].

This paper is organized as follows. In Section 2 I fix notation and review some aspects of 2D CFT of worldsheet orbifolds and their relation to open strings. In particular, the structure of possible group actions that generate open strings in these orbifolds models is elucidated. I also review briefly some basic aspects of the CS gauge theory on manifolds, in particular its connection with CFTs on closed oriented Riemann surfaces. In Section 2.2 it is shown how, upon looking for a 3D description of worldsheet orbifold CFTs, we are led to CS gauge theory on $\mathbb{Z}_2$ orbifolds. This allows us to make some preliminary conjectures about the correspondence between the spectra of these two theories.

These conjectures are confirmed in the remainder of the paper, where quantization of CS gauge theory on orbifolds is analyzed and a set of specific examples is given. In Section 3 I discuss the quantum CS gauge theory on orbifolds, first for arbitrary connected, simply connected gauge group G, and specializing to $G = \mathrm{SU}(2)$ afterwards. For any $\mathbb{Z}_2$ orbifold, the locus of all singular points comprises a link in the underlying topological manifold. Inside correlation functions, the singular locus is equivalent to a link of Wilson lines, which allows us to reduce the theory on orbifolds to a related theory on manifolds. This theory on manifolds is not necessarily the CS gauge theory with the same gauge group, as will be seen in detail in Section 3. The question of framing of the components of the singular locus, raised by their interpretation as a sum of Wilson lines, is studied briefly in Section 3.2. I complete the basic setting for the quantum theory on orbifolds in Section 3.3, where I discuss skein theory for the singular locus, and in Section 3.4, where the issue of observables is analyzed.

In the beginning of Section 4 I discuss the correspondence between CS gauge theory on $\mathbb{Z}_2$ orbifolds on the one hand, and 2D CFT of worldsheet orbifolds on the other. Most remarkably, the structure of Chan–Paton factors is elucidated (and fixed uniquely) within CS gauge theory in terms of the algebraic geometry of the singular locus. Sections 4 and 5 offer a set of basic examples that illustrate the correspondence. In Section 4 I study the CFT/CS gauge theory relation for SU(2), while in Section 5 the set of examples is extended to $c = 1$ CFTs (corresponding to $G = \mathrm{U}(1)$ CS gauge theory), and to holomorphic orbifold CFTs (CS gauge theory with discrete gauge groups). Those worldsheet orbifolds whose orbifold group mixes non-trivially the worldsheet parity transformation with a target action (the so-called "exotic worldsheet orbifolds") are shown to lead to an unusual form of gauge theory in 3D in which the orbifold group $\mathbb{Z}_2$ is mixed non-trivially with the CS gauge group. Possible implications of this phenomenon are discussed briefly in Section 5.3. Some elements of orbifold topology and geometry that are needed for the body of the paper are gathered in Appendix A; some more involved mathematical aspects of the definition of the Lagrangian for CS gauge theory on orbifolds with general gauge groups are deferred to Appendix B.

This paper is a rewritten version of a paper that was published in July 1990 as a Prague Institute of Physics preprint [12]. Although the results presented here are the same as in [12], the presentation has been altered. A part of the motivation for this revision (apart from the interest in the theory for reasons discussed above) comes from the possible applications this theory may have to boundary scattering in $1+1$-dimensional CFT. In fact, this recently very active area has a remarkably broad domain of applications, ranging from quantum impurity problems (such as the Kondo effect) to dissipative quantum mechanics, to propagation in quantum wires, to the Callan–Rubakov effect, to quantum theory of black holes. (See [13] and references therein for a review of most of these applications.) In many of these cases, the S matrix of the boundary scattering exhibits interesting properties [13,14] whose explanation, I believe, could come from the correspondence between 2D CFT on surfaces with boundaries and 3D CS theory on $\mathbb{Z}_2$ orbifolds as discussed in this paper. In fact, this correspondence suggests that the boundary scattering in 2D CFT can be alternatively described as an Aharonov–Bohm effect in 3D CS gauge theory. I hope to return to this point elsewhere.

2. Chern–Simons gauge theory on 3D orbifolds

CS gauge theory was formulated by Witten in [9] as a gauge theory in three dimensions with compact, connected and simply connected gauge group $\mathcal{G}$, and with the Lagrangian given by the CS functional: [2]

$$S(A) = \frac{k}{4\pi} \int_M \mathrm{Tr}\Big(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A\Big). \tag{2.1}$$

The set of observables of the theory is generated by Wilson lines

$$W_R(C) = \mathrm{Tr}_R\, P \exp \int_C A, \tag{2.2}$$

where R belongs to the finite set of integrable representations of the Kac–Moody algebra $\widehat{\mathcal{G}}$ at level k, and C is a closed line in M; and by "baryon" configurations first introduced in [15] and defined using trivalent vertices. At the quantum level, only a finite number of these vertices are relevant, corresponding to the information encoded in the structure constants of the fusion algebra of the associated WZW model.

Thus, the natural things to calculate are the correlation functions of the objects just mentioned:

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[2] Our normalization of S is such that the functional integral is weighted by e^{iS}.

$$\left\langle W_{R_1}(C_1)\, W_{R_2}(C_2)\cdots\right\rangle_M \equiv \int \mathrm{D}A\; W_{R_1}(C_1)\, W_{R_2}(C_2)\cdots \mathrm{e}^{\mathrm{i}S(A)}. \tag{2.3}$$

A particularly natural way of computing these correlation functions is the canonical quantization approach. To any surface Σ pierced in points z_i by Wilson lines in representations R_i there corresponds a (finite-dimensional) Hilbert space of quantum states, $\mathcal{H}_{\Sigma,R_i}$. Cutting the 3D manifold M into two parts along an orientable surface Σ, we can compute the amplitude as an inner product within $\mathcal{H}_\Sigma$, making use of the fact that the theory satisfies the axioms of topological QFT [16–18].

The key to the appeal of CS gauge theory for string physicists lies in the elegant relation of the theory to 2D rational CFT [9,19,20] (for a review of 2D CFT, see e.g. [21]). This correspondence identifies the Hilbert space of CS gauge theory canonically quantized on $\Sigma \times \mathbb{R}$, where Σ is a closed oriented surface, with the space of all conformal blocks of a rational CFT on Σ [9,19,20]. Since in this paper we are mainly interested in the correspondence between CS gauge theory and CFT, and this correspondence is well understood only for rational CFTs/compact CS gauge groups, we will restrict ourselves to rational CFTs throughout the paper, without mentioning the word "rational" explicitly.

2.1. *Conformal field theory of worldsheet orbifolds*

Our central interest throughout this paper is to reproduce 2D CFT of worldsheet orbifolds from CS gauge theory. I believe that a short review of the theory of worldsheet orbifolds may be useful. For other results not gathered here, see [4,7].

Let us choose a left–right symmetric CFT. Assume also that there is a discrete group $\widetilde{G}$ acting as a symmetry group on the theory in the target, i.e. exactly as in [6]. The theory is by assumption parity-invariant, i.e. there is a symmetry action of the worldsheet transformation

$$\Omega_0 : (z, \bar{z}) \mapsto (\mathrm{e}^{2\pi\mathrm{i}}\bar{z}, \mathrm{e}^{-2\pi\mathrm{i}}z) \tag{2.4}$$

on the fields of the theory. This particular action of the $\mathbb{Z}_2$ group on the 2D theory (i.e. the action that reverses the orientation of the worldsheet) plays a central role in the paper, and deserves a special notation; from now on, I will denote by $\mathbb{Z}_2^{\mathrm{ws}}$ this particular $\mathbb{Z}_2$ group generated by Ω_0 (or more precisely, by Ω, which is Ω_0 lifted trivially to the fields of the 2D theory).

Worldsheet orbifolds are then defined as orbifolds whose orbifold group G combines the worldsheet action of $\mathbb{Z}_2^{\mathrm{ws}}$ with a target symmetry given by $\widetilde{G}$, i.e.

$$G \subset \widetilde{G} \times \mathbb{Z}_2^{\mathrm{ws}}. \tag{2.5}$$

On worldsheet orbifolds, we can get essentially two distinct classes of twists. First, if G contains elements of the form $\tilde{g} \times 1$, where 1 is the identity of $\mathbb{Z}_2^{\mathrm{ws}}$ and $\tilde{g}$ is in $\widetilde{G}$, then usual twisted states are produced, exactly as in traditional (target) orbifold models. The other possibility, i.e. the case of twisting by an element acting non-trivially on the worldsheet by Ω, is a bit more intricate. In this case, we can easily observe that the choice of just one twisting element of G, say $g_1 \times \Omega$ (where g_1 is in $\widetilde{G}$), is not sufficient to fully determine the twisted state. If $g_1 \times \Omega$ corresponds to the twist of fields when we go around the cylindrical worldsheet in one direction,

$$\phi(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = (g_1 \times \Omega) \cdot \phi(z, \bar{z}) \equiv g_1 \cdot \phi(e^{2\pi i}\bar{z}, e^{-2\pi i}z), \tag{2.6}$$

we have to add another element, say $g_2 \times \Omega$, to determine the twist in the opposite direction:

$$\phi(e^{-2\pi i}z, e^{2\pi i}\bar{z}) = (g_2 \times \Omega) \cdot \phi(z, \bar{z}) \equiv g_2 \cdot \phi(e^{2\pi i}\bar{z}, e^{-2\pi i}z). \tag{2.7}$$

It is easy to show that (2.6) and (2.7) lead to open-string sectors [4].

This unusual structure of twisted states has a natural explanation if we think of the state twisted by the couple $g_1 \times \Omega$, $g_2 \times \Omega$ as an open-string state, with the open-string being a $\mathbb{Z}_2$ orbifold of the closed string. To specify twists on a particular worldsheet Σ, we have to specify monodromies of fields on Σ, i.e. a representation of the first homotopy group of Σ in the orbifold group:

$$\pi_1(\Sigma) \to G. \tag{2.8}$$

The open string is topologically an orbifold $\mathcal{O}_S$ of the closed string S^1 by $\mathbb{Z}_2$, $\mathcal{O}_S \equiv S^1/\mathbb{Z}_2$, and its orbifold fundamental group (see [1] for the definition) is $\mathbf{D}$, the infinite dihedral group:

$$\pi_1(\mathcal{O}_S) = \mathbf{D} \equiv \mathbb{Z}_2 * \mathbb{Z}_2 \equiv \mathbb{Z}_2 \ltimes \mathbb{Z}. \tag{2.9}$$

Here $*$ denotes the free product of groups, and $\ltimes$ is the semi-direct product. The monodromy of the open sector corresponds to a representation of the first homotopy group of the open string in the orbifold group:

$$\mathbb{Z}_2 * \mathbb{Z}_2 \to G, \tag{2.10}$$

required to satisfy one obvious geometrical constraint. The fundamental group of the open string, $\mathbb{Z}_2 * \mathbb{Z}_2$, is naturally mapped onto the group $\mathbb{Z}_2^{\text{ws}}$, both of its $\mathbb{Z}_2$ factors being mapped isomorphically to $\mathbb{Z}_2^{\text{ws}}$. Moreover, the orbifold group G has, as a natural subgroup in $\widetilde{G} \times \mathbb{Z}_2^{\text{ws}}$, a canonical projection onto $\mathbb{Z}_2^{\text{ws}}$. The worldsheet orbifold with the orbifold group G then admits only those representations (2.10) that complete the diagram

$$\mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2^{\text{ws}} \leftarrow G \tag{2.11}$$

to a commutative triangle. If G itself has the structure of a product:

$$G = G_0 \times \mathbb{Z}_2^{\text{ws}}, \tag{2.12}$$

the corresponding worldsheet orbifold will be referred to as a "standard" worldsheet orbifold. The complementary case, presumably more interesting, where the worldsheet group $\mathbb{Z}_2^{\text{ws}}$ mixed non-trivially with a target group action, is referred to as an "exotic" worldsheet orbifold.

At genus g, the partition function of a worldsheet orbifold CFT receives contributions from surfaces with h handles, b boundaries and c crosscaps, with $\frac{1}{2}b+\frac{1}{2}c+h=g$. On each particular surface Σ_g, the partition function contains the sum over all possible monodromies on Σ_g:

$$Z_{\Sigma_g}(m)=\frac{1}{|G|^g}\sum_{\alpha:\pi_1(\Sigma_g)\to G} Z_{\Sigma_g}(\alpha;m), \tag{2.13}$$

where m are the moduli, $\pi_1(\Sigma_g)$ is the (orbifold) fundamental group of Σ_g, and $Z_{\Sigma_g}(\alpha;m)$ denotes the amplitude calculated with the particular set of monodromies α. For exotic worldsheet orbifolds, the representations of $\pi_1(\Sigma_g)$ to be summed over are constrained analogously as in (2.11). $\pi_1(\Sigma_g)$ is a $\mathbb{Z}_2$ extension of the fundamental group of the double of Σ_g. Hence, there is a natural projection of $\pi_1(\Sigma_g)$ to $\mathbb{Z}_2^{\rm ws}$, and the allowed monodromies complete the following diagram,

$$\pi_1(\Sigma_g)\to\mathbb{Z}_2^{\rm ws}\leftarrow G, \tag{2.14}$$

to a commutative triangle. For example, the amplitude on the cylinder reads

$$Z_{\rm C}(t)=\frac{1}{|G|}\sum_{g_1,g_2,h} Z_{\rm C}(g_1,g_2,h;t), \tag{2.15}$$

where the monodromies are of the form $g_i=\tilde{g}_i\times\Omega$, $h=\tilde{h}\times 1$, as elements of $G\subset\widetilde{G}\times\mathbb{Z}_2^{\rm ws}$, and satisfy

$$g_1^2=g_2^2=1,\qquad [g_i,h]=1. \tag{2.16}$$

Much information about any theory is encoded in its one-loop amplitudes. In string theory, one-loop[3] diagrams correspond to genus-one topologies of the worldsheet; in unoriented open- and closed-string theory, they are given by the torus, Klein bottle, cylinder, and Möbius strip. The amplitudes can be computed in two different pictures [22]. The loop-channel picture corresponds to open and closed strings comprising loops of length t (with the width of the strings properly normalized). In this picture, the amplitudes can be calculated conveniently as traces over corresponding Hilbert spaces of closed and open strings. The tree-channel picture corresponds to a cylinder of length $\tilde{t}$ created from and annihilated to the vacuum via boundaries and crosscaps; the moduli t and $\tilde{t}$ of the two channels are related by $t=1/(2\tilde{t})$ for the Klein bottle and the Möbius strip, and by $t=2/\tilde{t}$ for the cylinder. It is well known [22] that the boundary and crosscap conditions on the fields can be translated into the quantum mechanical language by constructing the corresponding boundary and crosscap states $|B\rangle$, $|C\rangle$. This construction gives a simple recipe for calculating amplitudes in the tree channel. In the tree channel, the amplitude corresponds to the creation of a closed string from the vacuum by $\langle B|$ or $\langle C|$, subsequent free closed string propagation, and final annihilation into the vacuum by either $|B\rangle$ or $|C\rangle$.

[3] In the string coupling constant.

Comparing these two ways of computing the one-loop amplitudes we get a set of constraints:

$$\begin{aligned} \mathrm{Tr}_{\text{open}}\left(\mathrm{e}^{-H_{\mathrm{o}}t}\right) &= \langle B|\mathrm{e}^{-H_{\mathrm{c}}\tilde{t}}|B\rangle, \\ \mathrm{Tr}_{\text{open}}\left(\Omega\mathrm{e}^{-H_{\mathrm{o}}t}\right) &= \tfrac{1}{2}\left\{\langle B|\,\mathrm{e}^{-H_{\mathrm{c}}\tilde{t}}|C\rangle + \langle C|\,\mathrm{e}^{-H_{\mathrm{c}}\tilde{t}}|B\rangle\right\}, \\ \mathrm{Tr}_{\text{closed}}\left(\Omega\mathrm{e}^{-H_{\mathrm{c}}t}\right) &= \langle C|\,\mathrm{e}^{-H_{\mathrm{c}}\tilde{t}}|C\rangle, \end{aligned} \tag{2.17}$$

analogous to the requirements of modular invariance in closed CFT. The factor of one half in the middle equation of (2.17) is explained by observing that the $\mathbb{Z}_2$ symmetry that interchanges the two boundaries of the cylinder, or the two crosscaps of the Klein bottle, is to be divided out as a part of the gauge group in the full-fledged string theory, but not in CFT that we are considering here.

The one-loop conditions (2.17) pose stringent consistency restrictions on the theory. If we calculate amplitudes for a given model, say, in the loop channel, we must check whether corresponding boundary and crosscap states exist such that (2.17) be valid. Moreover, any relative normalization of the boundary state against the crosscap state, motivated e.g. from the BRST invariance in full-fledged string theory or from modular geometry in CFT, fixes the normalization of the loop-channel expressions. This normalization then self-consistently determines the Chan–Paton degeneration of the open sector of the string spectrum. This is an outline of how the Chan–Paton symmetry in open strings is controlled by modular geometry.

2.2. *A thickening of the open string*

In worldsheet orbifold models, left- and right-movers are coupled to each other through boundaries and/or non-orientability of the worldsheet. To find a correspondence of this coupling between left- and right-movers in the CS gauge theory, we have to identify how CFT with both left- and right sectors enters CS gauge theory. In the case of CFTs on closed oriented surfaces, an answer to this question was conjectured by Witten [23] and further developed by Moore et al. [19,20], Kogan [24], and Kogan and Carlip [25] (see also [26]). Their results can be simply summarized as follows.

Let us quantize the theory canonically on $C \times \mathbb{R}$, with C a cylinder [20]. Working in the axial gauge $A_0 = 0$, we must first satisfy the constraint that requires the space-like part of the curvature to be zero, $\widetilde{F} = 0$. This is easily solved to give (the tildes over $\widetilde{\mathrm{d}}$ and $\widetilde{A}$ denote the space-like parts of d and A):

$$\widetilde{A} = -\widetilde{\mathrm{d}}\widetilde{U}\,\widetilde{U}^{-1}, \qquad \widetilde{U} = U\exp\left(\mathrm{i}\frac{\lambda}{k}\phi\right), \tag{2.18}$$

where U is a single-valued map from C to $\mathcal{G}$, and λ measures the holonomy around the non-contractible loop on the cylinder. Inserting this solution into the Lagrangian (2.1), we can reduce it to an effective Lagrangian for U and λ:

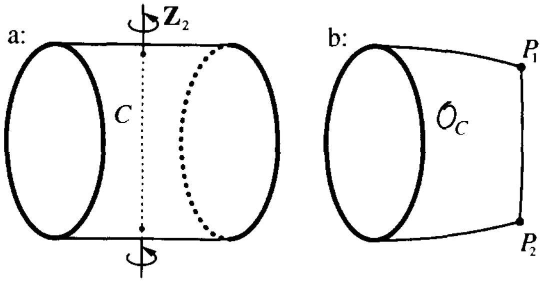

Fig. 1. (a) The $\mathbb{Z}_2$ symmetry of the cylinder C that defines the thickened open string $\mathcal{O}_C$ as $\mathcal{O}_C \equiv C/\mathbb{Z}_2$. (b) The thickened open string $\mathcal{O}_C$. The $\mathbb{Z}_2$ singular points P_1 and P_2 are the only singular points of $\mathcal{O}_C$.

$$S(U,\lambda) = \frac{k}{4\pi}\int_{\partial C\times\mathbb{R}} \mathrm{Tr}\left(U^{-1}\partial_\phi U U^{-1}\partial_t U\right)\,\mathrm{d}\phi\,\mathrm{d}t + \frac{k}{12\pi}\int_{C\times\mathbb{R}} \mathrm{Tr}\left(U^{-1}\,\mathrm{d}U\right)^3$$
$$+\frac{1}{2\pi}\int_{\partial C\times\mathbb{R}} \mathrm{Tr}\,\lambda(t)\left(U^{-1}\partial_t U\right)\,\mathrm{d}\phi\,\mathrm{d}t. \tag{2.19}$$

The Hilbert space $\mathcal{H}_C$ resulting from the quantization of this phase space has the structure of

$$\mathcal{H} = \bigoplus_\lambda [\phi_\lambda] \otimes \overline{[\phi_\lambda]}, \tag{2.20}$$

where λ now belongs to the set of integrable representations of the Kac–Moody group $\widehat{\mathcal{G}}$, and $[\phi_\lambda]$ denote the representations. This Hilbert space exactly corresponds [20] to the Hilbert space of the WZW model with $\widehat{\mathcal{G}}$ as its Kac–Moody symmetry group, and with a diagonal modular invariant. Within this correspondence, gauge invariant degrees of freedom living at one component of ∂C correspond to the left-movers, while the second component of ∂C yields the right-movers of the WZW CFT. Thus, the cylinder $C \equiv S^1 \times [0,1]$ is the manifold that represents the thickening of the closed string in CS theory, and similarly, $\Sigma \times [0,1]$ is the 3D thickening of closed oriented surface Σ.

Now we will look for an analogous 3D setting for open strings. In Section 2.1 we have seen in an outline how open strings emerge in twisted sectors of worldsheet orbifold models. Now I will argue that the orbifold construction extends also to the 3D CS theory: We will see that a natural thickening of the open string is a 2D $\mathbb{Z}_2$ orbifold with a boundary; I will also construct the thickened version of surfaces with boundaries and/or crosscaps, as particular 3D $\mathbb{Z}_2$ orbifolds. The final check of the proposed correspondence then comes from the fact that it reproduces the known structure of CFT on surfaces with boundaries and/or crosscaps (including such subtleties as the vacuum degeneration of the open string spectrum).

With this motivation in mind, we will proceed by studying CS gauge theory on $\mathbb{Z}_2$ orbifolds. To be a symmetry of the CS Lagrangian (2.1), the orbifold group $\mathbb{Z}_2$ must act on 3D "space–times" by orientation-preserving diffeomorphisms. A particularly important class of such actions are products of $\mathbb{R}$ (with the trivial $\mathbb{Z}_2$ action) and a 2D manifold Σ with an orientation-preserving involution. In particular, we may take $\Sigma = C$, the thickened

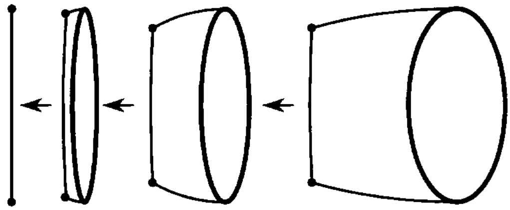

Fig. 2. The correspondence between the orbifold $\mathcal{O}_C$ and the open string $\mathcal{O}_S$. The singular points of $\mathcal{O}_C$ correspond to the boundary points of the open string, while the boundary of $\mathcal{O}_C$ corresponds to the interior of the open string worldsheet.

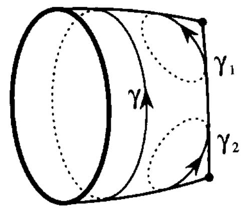

Fig. 3. The first homotopy group of $\mathcal{O}_C$, $\pi_1(\mathcal{O}_C) = \mathbb{Z}_2 * \mathbb{Z}_2 \equiv \mathbb{Z}_2 \otimes \mathbb{Z}$. The picture shows the generators γ_1, γ_2 of the $\mathbb{Z}_2$ components in $\mathbb{Z}_2 * \mathbb{Z}_2$, as well as their product γ that generates the normal subgroup $\mathbb{Z}$ in $\mathbb{Z}_2 \otimes \mathbb{Z}$.

closed string, and consider the $\mathbb{Z}_2$ action that interchanges the boundaries of C as in Fig. 1a. The resulting 2D orbifold, denoted by $\mathcal{O}_C$ throughout this paper, is the proposed thickened version of the open string (cf. Fig. 2).

One fact that supports this correspondence is the isomorphism between the first homotopy groups of the thickened open string $\mathcal{O}_C$ and the open string $\mathcal{O}_S$, which are both isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. For the thickened open string, the structure of the first homotopy group is indicated in Fig. 3. In particular, the generators of $\pi_1(\mathcal{O}_S)$ that correspond to the "boundary twists" on the open string (cf. (2.6) and (2.7)) now correspond to the non-contractible circles wrapped around the singular points of $\mathcal{O}_C$. (Actually, the "correspondence" of Fig. 2 is a homotopy equivalence in the corresponding category of orbifolds; cf. Appendix A.)

We have seen that any closed oriented worldsheet Σ of closed-string theory can be naturally thickened to a three-manifold $M = \Sigma \times [0, 1]$. As for surfaces of worldsheet orbifold models, i.e. surfaces with boundaries and/or crosscaps, we can construct their natural thickening as follows. Let Σ be a surface with boundaries and/or crosscaps, and $\overline{\Sigma}$ its oriented double with empty boundary. Denote by I the defining involution on $\overline{\Sigma}$, i.e. $\Sigma = \overline{\Sigma}/I$. The corresponding thickening of Σ is then

$$\mathcal{O}_\Sigma = (\overline{\Sigma} \times [0, 1])/I, \tag{2.21}$$

where I acts on $t \in [0, 1]$ via $t \to 1 - t$. $\mathcal{O}_\Sigma$ is an orbifold with boundary, $\partial\mathcal{O}_\Sigma$ being isomorphic to one component of $\overline{\Sigma}$. Two examples of such thickened open string diagrams are shown in Fig. 4.

At the quantum level, there is a correspondence between the partition function of the two-dimensional WZW model on a closed oriented surface Σ, and the (transition) amplitude of

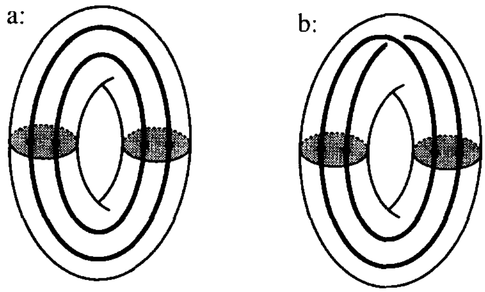

Fig. 4. The thickened versions of (a) the annulus, and (b) the Möbius strip diagrams. The thick lines represent the singular loci of the orbifolds; the shaded 2D sections are isomorphic to the thickened open string $\mathcal{O}_C$.

the CS gauge theory on M, summed up over the natural basis of $\mathcal{H}_\Sigma$:

$$Z_\Sigma = \sum h_{ij} \Psi_i \otimes \overline{\Psi}_j \in \mathcal{H}_\Sigma \otimes \overline{\mathcal{H}}_\Sigma. \tag{2.22}$$

The aforementioned correspondence between 2D surfaces of open string-theory and their 3D orbifold thickenings leads to an open-string counterpart of (2.22),

$$Z_\Sigma = \sum a_i \Psi_i \in \mathcal{H}_{\overline{\Sigma}}. \tag{2.23}$$

Here Σ is a surface with at least one boundary or crosscap, and $\overline{\Sigma}$ denotes its double.

This picture allows us to make, already at this stage, some preliminary conjectures about the relation between 3D CS gauge theory on orbifolds and 2D CFT of open strings. The closed string, which is topologically a circle, can be obtained from its thickening if the boundaries of the thickening approach each other. In the case of open strings, the analogous procedure of shrinking the thickening $\mathcal{O}_C$ to the open string is shown in Fig. 2. The boundary points of the string correspond to the two points in the singular locus of the thickening. We thus expect that the structure of Chan–Paton factors is related to the geometry of the singular locus. On the other hand, the bulk degrees of freedom on the worldsheet of the open string are expected to correspond to gauge-invariant degrees of freedom at the boundary of the thickening. These expectations will be confirmed in Section 3.

3. Quantization of CS gauge theory on orbifolds

We have seen in Section 2 that the natural setting for the CS counterpart of 2D CFT on surfaces with boundaries and crosscaps is the theory on three-dimensional $\mathbb{Z}_2$ orbifolds, and we have made several preliminary conjectures about the correspondence between these two theories. In order to substantiate these expectations, we must quantize the CS theory on orbifolds and compare the outcome to the structure of 2D CFT of worldsheet orbifolds.

As a first step towards the definition of the quantum CS gauge theory with gauge group $\mathcal{G}$ on an orbifold, we have to specify a Lagrangian for connections on any principal $\mathcal{G}$-bundle over arbitrary orbifold $\mathcal{O}$. One is tempted to define, in analogy with string theory on orbifolds, the Lagrangian on an orbifold $\mathcal{O}$ for a given $\mathcal{G}$-bundle E via the Lagrangian of CS gauge theory on the doubling $\overline{\mathcal{O}}$ of $\mathcal{O}$:

$$2S(A) = S(\overline{A}) \equiv \frac{k}{4\pi} \int_{\overline{\mathcal{O}}} \mathrm{Tr}\Big(\overline{A} \wedge \mathrm{d}\overline{A} + \tfrac{2}{3}\overline{A} \wedge \overline{A} \wedge \overline{A}\Big). \tag{3.1}$$

Here $\overline{A}$ is the pullback of the connection A from $\mathcal{O}$ to the doubling bundle $\overline{E}$ over the doubling manifold $\overline{\mathcal{O}}$. The formula is well defined at least for compact, connected, simply connected gauge groups, $\overline{E}$ being in this case the trivial principal bundle over the manifold $\overline{\mathcal{O}}$. Nevertheless, this definition is still incomplete, since we have to resolve the ambiguity that has emerged because we have defined a *multiple* of $S(A)$ in (3.1), and we have to resolve this ambiguity for any orbifold in a way compatible with factorization (see [18] for a thorough discussion of this argument in a slightly different context). For general gauge groups, this requires techniques of equivariant cohomology of classifying spaces, and I refer the reader to Appendix B, where the general answer is presented.

In order to avoid technical complications, I will now focus on $\mathcal{G}$ compact, connected, and simply connected. The classical phase space to be canonically quantized on a Hamiltonian slice Σ of a 3D "space–time", is given by

$$\mathcal{P}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathcal{G}) \times \mathrm{Maps}(\partial\Sigma, \mathcal{G}). \tag{3.2}$$

One basic building block of this phase space is the space of possible holonomies around a singular point. Ignoring temporarily the overall conjugation by $\mathcal{G}$, it is given by all representations of $\mathbb{Z}_2$ in $\mathcal{G}$, i.e. the submanifold $\mathcal{G}_{(2)}$ in $\mathcal{G}$ of those elements whose square is one. The phase space $\mathcal{G}_{(2)}$ pierces a fixed maximal torus $\mathcal{T}$ in a finite set $\mathcal{T}_{(2)}$, and in turn, $\mathcal{G}_{(2)}$ can be recovered from this finite set by conjugating $\mathcal{T}_{(2)}$ by $\mathcal{G}$. Thus, $\mathcal{G}_{(2)}$ can be decomposed into conjugacy classes $\omega \mathrm{e}^{\lambda}\omega^{-1}$ classified by $\mathrm{e}^{\lambda} \in \mathcal{T}_{(2)}$.

Let us specialize for simplicity to $\Sigma = \mathcal{O}_{\mathrm{D}}$, the disk with one singular point inside, and define the CS Lagrangian on the unconstrained phase space, restricting ourselves to the holonomies that are conjugated to a particular element e^{λ} of $\mathcal{T}_{(2)}$. The general case can be treated similarly. Respecting all the required symmetries, we get the Lagrangian that combines the usual CS Lagrangian with the coadjoint orbit Lagrangian for the holonomies around the singular point:

$$\begin{aligned} S(A,\omega) = {} & \frac{k}{8\pi} \int_{\overline{\mathcal{O}}_D} \mathrm{Tr}\left(\overline{A} \wedge \mathrm{d}\overline{A} + \tfrac{2}{3}\overline{A} \wedge \overline{A} \wedge \overline{A}\right) \\ & + \int \mathrm{d}t\, \mathrm{Tr}\left(\lambda\omega^{-1}(t)\,(\partial_t + A_0)\,\omega(t)\right), \end{aligned} \tag{3.3}$$

with the notation of (3.1), and ω parametrizing the component of $\mathcal{G}_{(2)}$ consisting of the elements conjugated to e^{λ}. This Lagrangian is anomalous unless λ is a weight [20]. This condition poses restrictions on possible values of k (cf. [20]); I will limit the discussion henceforth to the non-anomalous k's. Consequently, quantization of the corresponding effective Lagrangian on the constrained phase space $\mathcal{P}$ leads to the Hilbert space consisting of irreducible representations $[\phi_\lambda]$ of the loop group $\mathcal{LG}$, with λ being from $\mathcal{T}_{(2)}$. In this section I will only discuss the simplest case of $\mathcal{G} = \mathrm{SU}(2)$, unless stated otherwise.

On our favorite orbifold $\mathcal{O}_C$, the reduced phase space $\mathcal{P}$ consists of the product of two copies of $\mathcal{G}_{(2)}$, times the space of gauge-invariant degrees of freedom that survive at the boundary. In the case of $\mathcal{G} = \mathrm{SU}(2)$, the set of holonomies allowed around the singular points reduces to two points, corresponding to the representations of spin 0 and $\frac{1}{2}k$ at level k, which is then necessarily even. Quantization of the corresponding phase space leads to the Hilbert space

$$\mathcal{H} = 2\left\{[\phi_0] \oplus [\phi_{k/2}]\right\}. \tag{3.4}$$

According to the correspondence between CS gauge theory and two-dimensional CFT of open strings, we expect this space to represent the Hilbert space of open string states of a worldsheet orbifold of the SU(2) WZW model. I will demonstrate that this is indeed the case in Section 4.1, where I identify explicitly the CFT that corresponds to (3.4).

3.1. The singular locus as a link of Wilson lines

We have seen in (3.3) that the points in which the singular locus pierces a chosen Hamiltonian slice effectively behave as sources of curvature for the CS gauge field. More precisely, the form of the Lagrangian (3.3) indicates that the singular locus is effectively equivalent in the quantum theory to a sum of Wilson lines in some particular representations of the gauge group. This simple but important fact allows us to reduce the theory on orbifolds to a theory on manifolds, trading the singular locus for a link of Wilson lines.

Another argument that will allow us to see the equivalence, follows closely the reasoning of [15]. Consider a connected component of the singular locus in a 3D orbifold M, and denote it by ℓ. It can be surrounded by a 2D torus T, which divides M into two disconnected parts, i.e. a solid torus with ℓ inside it, and the remnant. Whatever happens inside the solid torus, defines a vector from $\mathcal{H}_T$. A natural basis in $\mathcal{H}_T$ is given by functional integrals over the solid torus with all the allowed Wilson lines replacing ℓ. The vector that describes the functional integral with the component ℓ of the singular locus inside the solid torus can be expanded in this basis,

$$\ell = \sum_{R_i} c_{R_i} W_{R_i}(\ell), \tag{3.5}$$

where c_{R_i} is a set of complex numbers. Effectively, all information about the presence of orbifold singularities is now stored in these numbers.

We have just argued that any connected component ℓ of the singular locus on an orbifold $\mathcal{O}$ can be represented as a sum over Wilson lines with the topology of ℓ. As a result of this equivalence, the theory on orbifolds is reduced to the "parent" theory on manifolds,[4] as

[4] In some more complicated cases, discussed in Sections 4.2 and 5, the gauge group of the "parent" theory may differ from $\mathcal{G}$ by a discrete factor.

follows. Using (3.5), the correlation function of an arbitrary collection of physical observables Φ on $\mathcal{O}$ can be calculated as a sum over more complicated correlation functions on the underlying manifold $X_{\mathcal{O}}$,[5] with the singular locus traded for a link of specific Wilson lines:

$$\begin{aligned}\langle\Phi\rangle_{\mathcal{O}} &= \left\langle \Phi \prod_{\alpha} \left(\sum_{R_i} c_{R_i} W_{R_i}(\ell_\alpha) \right) \right\rangle_{X_{\mathcal{O}}} \\ &\equiv \sum_{R_i^{(1)},\ldots,R_j^{(s)}} c_{R_i^{(1)}} \ldots c_{R_i^{(s)}} \langle \Phi\, W_{R_i^{(1)}}(\ell_1) \ldots W_{R_i^{(s)}}(\ell_s) \rangle_{X_{\mathcal{O}}}. \end{aligned} \tag{3.6}$$

(Here $\alpha = 1, \ldots, s$ counts the connected components ℓ_α of the singular locus, and $X_{\mathcal{O}}$ is the underlying manifold of $\mathcal{O}$.) In particular, the partition function of the CS gauge theory on the orbifold is equivalent to a correlation function of the usual CS gauge theory on the underlying manifold:

$$\begin{aligned} Z(\mathcal{O}) &= \left\langle \prod_{\alpha} \left(\sum_{R_i} c_{R_i} W_{R_i}(\ell_\alpha) \right) \right\rangle_{X_{\mathcal{O}}} \\ &\equiv \sum_{R_i^{(1)},\ldots,R_j^{(s)}} c_{R_i^{(1)}} \ldots c_{R_i^{(s)}} \langle W_{R_i^{(1)}}(\ell_1) \ldots W_{R_i^{(s)}}(\ell_s) \rangle_{X_{\mathcal{O}}}. \end{aligned} \tag{3.7}$$

These two formulas represent one of the central points of this paper. They relate the correlation function in a theory on orbifolds (which we a priori do not know how to calculate) to a sum of more complicated correlation functions in the simpler theory on manifolds (which we do know how to calculate).

To establish the correspondence between the theory on orbifolds and the theory on the underlying manifolds, it now only remains to determine the c_{R_i}'s of (3.5). To this aim let us consider the theory on an orbifold which is topologically a solid torus, with the singular locus isomorphic to the generator of the fundamental group. This functional integral determines a state from the Hilbert space on the torus. We can measure this state by the following procedure. Let us take another copy of the solid torus, now with an arbitrary Wilson line $W_R(b)$ replacing ℓ, with $b \sim \ell$ topologically, and glue these two solid tori together, so as to obtain $S^2 \times S^1$. The functional integral of the resulting object is easily calculable as a trace over the physical Hilbert space of the twice punctured sphere. On the other hand, the same amplitude is equal to the inner product of the states that result from functional integrals over the solid tori before gluing. This leads to the following formula, which allows one to determine c_{R_i}:

$$Z(S^2 \times S^1, R, \ell) = \sum_{R_i} c_{R_i} \left(v_{R_i}, v_R \right). \tag{3.8}$$

[5] $X_{\mathcal{O}}$ is topologically the same as $\mathcal{O}$ but with orbifold singularities smoothed out. We are safe here, at least if $X_{\mathcal{O}}$ is a topological manifold, because every 3D topological manifold admits exactly one differentiable structure.

Here (,) denotes the inner product in $\mathcal{H}_T$, and R_i are the representations carried by the singular locus. This completes the arguments on the equivalence (3.5) between the singular locus and a sum of Wilson lines with the same topology. As a sample application of this equivalence, note that each component of the singular locus in the SU(2) theory is equivalent to $W_0(C) + W_{k/2}(C)$.

3.2. Framing of the singular locus

It is known that in quantum CS gauge theory, Wilson lines need framing. In particular, the singular locus, being equivalent to a sum of links of Wilson lines, may need framing. One may thus wonder how the statements of the previous paragraph interfere with this additional structure needed for a well-defined quantum theory.

First of all, note that the singular locus of any orbifold required for the correspondence to 2D CFT can be canonically framed. We have seen in Section 2.2 that the orbifolds representing the thickening of an open-string surface are of the form

$$\mathcal{O} = (\Sigma \times [0, 1])/\mathbb{Z}_2. \tag{3.9}$$

Such an orbifold can indeed be retracted uniquely (up to homotopy) to the 2D surface $\Sigma/\mathbb{Z}_2$. Thus, we can pick an arbitrary imbedding of the 2D surface $\Sigma/\mathbb{Z}_2$ into $\mathcal{O}$, with boundaries mapped to the singular locus of $\mathcal{O}$. This retraction gives a unique and natural framing to the singular locus, simply by demanding that the vectors that frame the singular locus are tangent to the image of $\Sigma/\mathbb{Z}_2$, and point inward.

Since the canonical framing of the singular locus always exists (and is unique) for the orbifolds that represent the thickened open-string surfaces, we need not worry about framing in the applications to 2D CFT on surfaces with boundaries and/or crosscaps; we would still need something more, however, were we interested in the full-fledged CS gauge theory on general $\mathbb{Z}_2$ orbifolds. Results of [27], which indicate that there might be a preferred way how to frame a three-manifold, are particularly interesting in this context. Alternatively, we could restrict ourselves to those models that do not require framing of the singular locus, i.e. do not require framing of the particular set of Wilson lines that effectively represent the singular locus in the correlation functions according to (3.5). This restriction would impose an additional condition on the CS coupling constant k. For example, in the case of $G = \mathrm{SU}(2)$ that we have been focusing on in this section, the singular locus carries the representations with spin 0 or $\frac{1}{2}k$. If the framing of a Wilson line $W_R(C)$ is shifted by a t-fold twist, the corresponding state is multiplied by $\mathrm{e}^{2\pi \mathrm{i} h_R t}$, where h_R is the conformal weight of the primary field corresponding to R. Conformal weights of the primaries ϕ_j of the SU(2) WZW model are

$$h_j = j(j+1)/(k+2); \tag{3.10}$$

hence, the conformal weight of the non-trivial primary $\phi_{k/2}$ carried by the singular locus of the SU(2) theory equals $h_{k/2} = \frac{1}{4}k$. Insisting on the integrality of the conformal weights of the primaries that correspond to the singular locus, we get the restriction $k = 0 \pmod 4$ on the coupling constant of the CS gauge theory on orbifolds.

3.3. Skein theory for the singular locus

One of the most appealing and important properties of CS correlation functions of Wilson lines is their calculability by (un)braiding the Wilson lines using skein theory. To a given two-dimensional surface Σ with p punctures and representations R_i, $i = 1, \ldots, p$ inserted in them, CS gauge theory assigns the Hilbert space $\mathcal{H}_{\Sigma,R_i}$ of physical states, which is n-dimensional. The skein relations are linear dependence relations, satisfied by any set of $n + 1$ vectors of this vector space.

This braiding procedure plays an interesting role in the comparison with CFTs of worldsheet orbifolds. Indeed, the only difference between the thickened cylinder (Fig. 4a) and the thickened Möbius strip (Fig. 4b) is in braiding of the singular locus. More explicitly, the functional integral over these two topologies, with the particular labeling of the Wilson lines, gives the partition functions of the associated 2D CFT on the worldsheet of the topology of the cylinder and Möbius strip respectively, Ψ_{C} and Ψ_{MS}, which are elements of the Hilbert space of the CS gauge theory on the torus. Using the results of Sections 2 and 3.2, we have

$$\Psi_{\mathrm{C}} = \sum_{R,R' \in \{0,k/2\}} \langle W_R(\ell) W_{R'}(\ell') \rangle_{\text{solid torus}} \tag{3.11}$$

and

$$\Psi_{\mathrm{MS}} = \sum_{R \in \{0,k/2\}} \langle W_R(\tilde{\ell}) \rangle_{\text{solid torus}}. \tag{3.12}$$

Here ℓ, ℓ', $\tilde{\ell}$ denote components of the singular loci as shown in Fig. 4. The only difference between the two orbifolds can be localized within a small two-sphere, pierced four times by the singular locus. Cutting out the ball surrounded by this two-sphere, we get an orbifold $\mathcal{O}$ whose boundary ∂M is isomorphic to the disconnected sum of the torus and the four punctured sphere. Then we can compute the Ψ_{C} and Ψ_{MS} of (3.11) and (3.12) as inner products in the Hilbert space of the four-times punctured sphere $\mathcal{H}_{S^2}$:

$$\Psi_{\mathrm{C}} = (u, v), \qquad \Psi_{\mathrm{MS}} = (u, v'), \tag{3.13}$$

where $u \in \mathcal{H}_{S^2}$ represents the functional integral over $\mathcal{O}$, and v, v' are the functional integrals over the three-balls with Wilson lines as shown in Fig. 5b.

Let us now restrict ourselves to $\mathcal{G} = \mathrm{SU}(2)$ with $k = 0 \pmod 4$, for which we have seen in Section 3.2 that the theory is independent of framing of the singular locus. The singular locus is equivalent to a sum of Wilson lines with $R, R' \in \left\{0, \frac{1}{2}k\right\}$. For such R, R', the corresponding Hilbert space is one-dimensional, as can be easily inferred from the fusion rules of the SU(2) WZW model [21]:

$$[\phi_{j_1}] \times [\phi_{j_2}] = \sum_{j=|j_1-j_2|}^{j=\min(j_1+j_2,\, k-j_1-j_2)} [\phi_j], \qquad j_1, j_2, j \in \left\{0, \tfrac{1}{2}, \ldots \tfrac{1}{2}k\right\}. \tag{3.14}$$

Thus, any two states of the physical Hilbert space are linearly dependent. In particular, with our restriction on k, the vectors given by the functional integrals over the three-dimensional

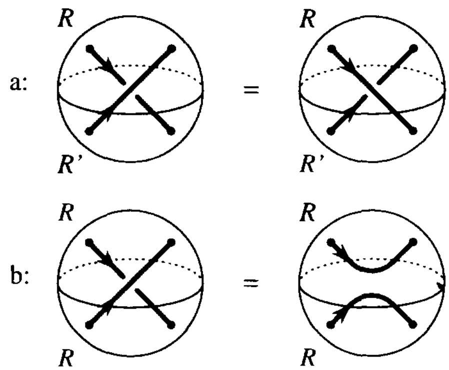

Fig. 5. Skein relations of the singular locus. R and R' are the SU(2) representations that can be carried by the singular locus, i.e. their spins are either 0 or $\frac{1}{2}k$, and k is assumed to satisfy $k = 0 \pmod 4$).

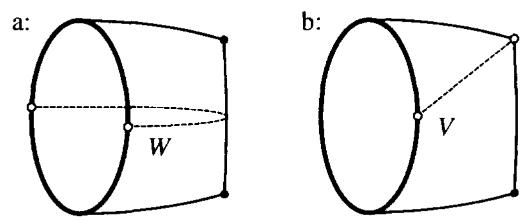

Fig. 6. The topology of observables in a hamiltonian slice of the thickened open string. The Wilson line denoted by the dashed line corresponds to (a) the thickened closed string vertex operator, (b) the thickened open string vertex operator.

balls with the Wilson lines as in Fig. 5a are equal to each other, the same being true of the amplitudes in Fig. 5b. As a consequence, the action of the braid group on the singular points of $\mathcal{O}_C$ reduces to the action of the permutation group, and any multiple twist can be trivially unbraided.

3.4. Observables

Since the CS gauge theory on orbifolds can be effectively reduced to a CS gauge theory on manifolds, observables on orbifolds are of precisely the same structure as those on manifolds (with an obvious orbifold-like projection included). In particular, the Wilson lines indicated in Fig. 6 are natural candidates for observables. Note that, using the equivalence (3.6) of the singular locus and a link of Wilson lines, we can interpret the observable V of Fig. 6b as a trivalent graph, with the remaining two legs corresponding to the singular locus that pierces the hamiltonian slice at the endpoint of V.

In the closed-string case, CS counterparts of 2D vertex operators were identified by Kogan and Carlip [24,25] with the Wilson lines going from one boundary of a thickened worldsheet $\Sigma \times [0, 1]$ to the other (possibly with some gauge invariant quantities attached at their ends). Indeed, these Wilson lines transform under a gauge transformation g as the

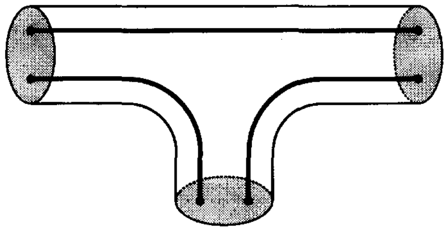

Fig. 7. The 3D orbifold that represents the thickened version of the open-string interaction.

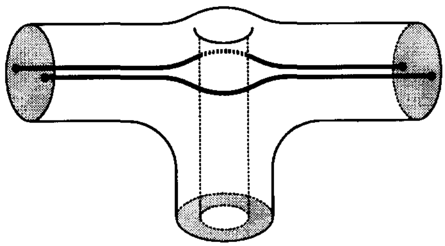

Fig. 8. The thickened version of the open-string/closed string interaction.

product of one left-moving and one right-moving Kac–Moody primary of the WZW CFT model:

$$W \to g(x) \cdot W \cdot g(y)^{-1}, \tag{3.15}$$

where x and y are the endpoints of the Wilson line.

The thickened version of the open-string three-vertex is the "solid pant" orbifold of Fig. 7. It represents an interpolation between three thickened open strings $\mathcal{O}_C$. (To get rid of the $\mathbb{Z}_2$ odd states in the open-string sector, we should sum over all possible permutations of the ends of single components of the singular locus.) With the use of (3.7), the singular locus can be traded for a sum of Wilson lines, hence the transition amplitudes between two-dimensional orbifolds are in principle computable as a non-trivial scattering problem on the underlying manifolds! This is an interesting CS incarnation of the old idea that Chan–Paton charges at the ends of strings represent dynamical particles (quarks of the old dual models). The necessity of summation over all permutations in this scattering process resembles the analogous statements in the gravitational scattering in 2 + 1 dimensions [28].

As for the closed string/open string interaction, its thickened version is shown in Fig. 8. At the level of fundamental groups, the interaction is equivalent to an action of the fundamental group of the thickened closed string C on the fundamental group of the thickened open string $\mathcal{O}_C$. To see this more explicitly, let us denote by Γ the generator of $\pi_1(C) = \mathbb{Z}$, and by γ_1, γ_2 (respectively γ_1', γ_2') generators of the two $\mathbb{Z}_2$ components of $\pi_1(\mathcal{O}_C) = \mathbb{Z}_2 * \mathbb{Z}_2$ before (respectively after) the interaction. Then the interaction acts on $\pi_1(\mathcal{O}_C)$ as follows:

$$(\gamma_1, \gamma_2) \to (\gamma_1', \gamma_2') = (\gamma_1, \Gamma\gamma_2\Gamma^{-1}). \tag{3.16}$$

With this picture of thickened string interactions now at hand, it is not too complicated to see that the infinitesimal versions of the interactions shown in Figs. 7 and 8 are the Wilson lines of Fig 6. In particular, the open-string emission from an open string is represented in CS gauge theory by a trivalent-graph observable.

4. The Chern–Simons/CFT correspondence and first examples

The relation between CS gauge theory on orbifolds and CFT of worldsheet orbifolds leads to a surprisingly simple prescription for the open-string spectrum of a 2D worldsheet orbifold CFT, including its Chan–Paton degeneration, once the corresponding CS gauge group is known.

The prescription can be briefly sketched as follows. Given a worldsheet orbifold CFT, consider first the restriction of the theory to closed oriented surfaces. Next, identify the CS gauge group that is associated with this "parent" CFT on closed oriented surfaces, in the classification of [19]. To get from this "parent" CS theory on manifolds to a theory on $\mathbb{Z}_2$ orbifolds, we must identify which class of holonomies is allowed around the singular locus. To this goal, in analogy with the quantization of (3.3), we identify the primaries that correspond to those elements of the gauge group that square to one. Let us denote the set of these primaries by $\mathcal{R} = \{\phi_r,\ r = 1, \ldots, N\}$. These primaries are examples of the boundary states in the 2D model on surfaces with boundaries and/or crosscaps, described first by Cardy in [29] (cf. the states denoted by $|\tilde{I}\rangle$ in [29, Eq. (21)]). To obtain the spectrum of the open sector, we label each of the two singular points of the thickened open string by a representation from $\mathcal{R}$. Fusion rules of these representations give the bulk part of the open-string spectrum, while the structure constants of the fusion algebra determine the Chan–Paton degeneration of the states. The whole spectrum results from all possible combinations of the labeling of the singular points by elements of $\mathcal{R}$.

In this form, the correspondence between CS gauge theories on $\mathbb{Z}_2$ orbifolds and CFTs of worldsheet orbifolds clearly does not cover all possible CFTs. Generically, we can associate several different sets of boundary and crosscap conditions to a given CFT on closed oriented surfaces; yet, the correspondence that we have just outlined associates one preferred set of boundary and crosscap conditions to each CS gauge group (and hence, to the corresponding CFT on closed oriented surfaces). The question is whether the other types of boundary conditions can also be incorporated into the scheme. This question will be answered in the affirmative later in the paper, after we learn more by studying several explicit examples of the correspondence between the CS gauge theory on $\mathbb{Z}_2$ orbifolds and CFT on surfaces with boundaries and/or crosscaps. It turns out that the general scheme will require an extension of the standard definition of CS gauge theory that will incorporate the orbifold group $\mathbb{Z}_2$ into the gauge group.

4.1. SU(2) *CS gauge theory and worldsheet orbifolds*

I conjectured in the beginning of Section 3 that the Hilbert space of the SU(2) CS gauge theory on the thickened open string (as given by (3.4)) should correspond to the open-string sector of a 2D CFT. Now I will identify this two-dimensional CFT.

Inspired by (3.4), we are looking for a worldsheet orbifold of the SU(2) WZW model that has only two primaries in the twisted (i.e. open) part of the spectrum (modulo a possible Chan–Paton degeneracy), namely $\phi_0 \equiv 1$ and $\phi_{k/2}$. Theory of open strings on group manifolds was studied by Ishibashi in [30]. In fact, Ishibashi constructed one particular SU(2)-model for each k even. These models are examples of worldsheet orbifold models as discussed in Section 2.1. Ishibashi starts with a diagonal modular invariant, and takes for the projection operator in the closed sector the parity operator acting in the obvious way on the Kac–Moody algebra, and trivially on the basis of primary states:

$$\Omega J_n^a \Omega^{-1} = \overline{J}_n^a, \qquad \Omega \left| j \otimes \overline{j} \right\rangle = \left| \overline{j} \otimes j \right\rangle. \tag{4.1}$$

This definition specifies uniquely the Klein bottle amplitude of the model, which in the loop channel reads

$$Z_{\mathrm{KB}}(t) = \sum_{j \in (1/2)\mathbb{Z}_k} \chi_j(2\mathrm{i}t). \tag{4.2}$$

Using the form of the S matrix for the SU(2) WZW model:

$$S_j^l = \sqrt{\frac{2}{k+2}} \sin \frac{(2j+1)(2l+1)\pi}{k+2}, \tag{4.3}$$

we can transform the amplitude to the tree channel,

$$Z_{\mathrm{KB}}(\tilde{t}) = \sum_{j,l \in (1/2)\mathbb{Z}_k} S_j^l \chi_l(2\mathrm{i}t) = \sum_{j \in \mathbb{Z}_{k/2}} \sqrt{\frac{2}{k+2}} \cot \frac{(2j+1)\pi}{2(k+2)} \chi_j(\mathrm{i}\tilde{t}), \tag{4.4}$$

and infer from these formulas the form of the full crosscap state of the model:

$$|C\rangle = \sum_{j \in \mathbb{Z}_{k/2}} \sqrt[4]{\frac{2}{k+2}} \cot^{1/2} \left(\frac{(2j+1)\pi}{2(k+2)} \right) |C, j\rangle, \tag{4.5}$$

where $|C, j\rangle$ are normalized so as to give the corresponding character,

$$\langle C, j | \, C, l \rangle = \delta_{jl} \chi_j. \tag{4.6}$$

The open-string part of the spectrum is then required to satisfy constraints (2.17) that embody modular properties of the model. Ref. [30] shows that open strings carrying each of the integer-spin integrable representations at level k is a good choice. With the projection operator in the open sector acting trivially on the primary states, $\Omega \, |j\rangle = |j\rangle$, the following amplitudes can be computed:

$$\begin{aligned} Z_{\mathrm{C}}(t) &= \sum_{j \in \mathbb{Z}_{k/2}} \chi_j(\tfrac{1}{2}\mathrm{i}t) = \sum_{j \in \mathbb{Z}_{k/2}, l \in (1/2)\mathbb{Z}_k} S_j^l \chi_l(\mathrm{i}\tilde{t}) \\ &= \sqrt{\frac{2}{k+2}} \sum_{j \in \mathbb{Z}_{k/2}} \sin^{-1} \left(\frac{(2j+1)\pi}{k+2} \right) \chi_j(\mathrm{i}\tilde{t}), \\ Z_{\mathrm{MS}}(t) &= \sum_{j \in \mathbb{Z}_{k/2}} \chi_j(\tfrac{1}{2} + \tfrac{1}{2}\mathrm{i}t) = \sum_{j \in \mathbb{Z}_{k/2}, l \in \frac{1}{2}\mathbb{Z}_k} M_j^l \chi_l(\tfrac{1}{2} + \mathrm{i}\tilde{t}) \end{aligned} \tag{4.7}$$

$$= \frac{1}{\sqrt{k+2}} \sum_{j \in \mathbb{Z}_{k/2}} \sin^{-1}\left(\frac{(2j+1)\pi}{2(k+2)}\right) \chi_j(\tfrac{1}{2} + \mathrm{i}\tilde{t}),$$

where the Möbius strip diagram is transformed to the tree channel by the modular transformation M acting on the characters as

$$M^i_j = \begin{cases} \dfrac{2}{\sqrt{k+2}} \sin \dfrac{(2j+1)(2i+1)\pi}{2(k+2)} & \text{for } i+j \in \mathbb{Z}, \\ 0 & \text{for } i+j \in \mathbb{Z} + \tfrac{1}{2}. \end{cases} \tag{4.8}$$

These amplitudes satisfy requirements (2.17), the boundary state of this model being

$$|B\rangle = \sum_{j \in \mathbb{Z}_{k/2}} \sqrt[4]{\frac{2}{k+2}} \sin^{-1/2}\left(\frac{(2j+1)\pi}{(k+2)}\right) |B, j\rangle, \tag{4.9}$$

with the states $|B, j\rangle$ defined analogously as the $|C, j\rangle$.

Ishibashi's model was obtained as an orbifold model using the $\mathbb{Z}_2$ action of (4.1) on the conventional SU(2) WZW model. Nevertheless, this worldsheet orbifold model is not equivalent to the model we are searching for, since its open-string spectrum does not correspond to the spectrum of (3.4) obtained in the SU(2) CS gauge theory on orbifolds.

The 2D CFT that does correspond to the SU(2) CS gauge theory on orbifolds as discussed in Section 3 can be identified by inverting the strategy that we used above when we derived the cylinder and Möbius strip amplitudes of Ishibashi's model. Starting from the "Chern–Simons inspired" spectrum (3.4), after some algebra we obtain the one-loop amplitudes corresponding to this spectrum:

$$\begin{aligned} Z_{\mathrm{C}}(t) &= 2\chi_0(t) + 2\chi_{k/2}\left(\tfrac{1}{2}\mathrm{i}t\right) = 2 \sum_{j \in \{0, k/2\}, l \in (1/2)\mathbb{Z}_k} S^l_j \chi_l(\mathrm{i}\tilde{t}) \\ &= 4\sqrt{\frac{2}{k+2}} \sum_{j \in \mathbb{Z}_{k/2}} \sin \frac{(2j+1)\pi}{k+2} \chi_j(\mathrm{i}\tilde{t}), \\ Z_{\mathrm{MS}}(t) &= 2\chi_0\left(\tfrac{1}{2} + \tfrac{1}{2}\mathrm{i}t\right) = \sum_{l \in \frac{1}{2}\mathbb{Z}_k} M^l_0 \chi_l\left(\tfrac{1}{2} + \mathrm{i}\tilde{t}\right) \\ &= \frac{2}{\sqrt{k+2}} \sum_{j \in \mathbb{Z}_{k/2}} \sin\left(\frac{(2j+1)\pi}{2(k+2)}\right) \chi_j(\tfrac{1}{2} + \mathrm{i}\tilde{t}). \end{aligned} \tag{4.10}$$

The boundary and crosscap states corresponding to these amplitudes are easily found to be:

$$\begin{aligned} |B\rangle &= \sum_{j \in \mathbb{Z}_{k/2}} 2\sqrt[4]{\frac{2}{k+2}} \sin^{1/2}\left(\frac{(2j+1)\pi}{(k+2)}\right) |B, j\rangle, \\ |C\rangle &= \sum_{j \in \mathbb{Z}_{k/2}} \sqrt[4]{\frac{2}{k+2}} \tan^{1/2}\left(\frac{(2j+1)\pi}{2(k+2)}\right) |C, j\rangle, \end{aligned} \tag{4.11}$$

which leads to the Klein bottle amplitude:

$$\begin{aligned} Z_{\mathrm{KB}} &= \sqrt{\frac{2}{k+2}} \sum_{j\in\mathbb{Z}_{k/2}} \tan\frac{(2j+1)\pi}{2(k+2)}\, \chi_j(\mathrm{i}\tilde{t}) \\ &= \sqrt{\frac{2}{k+2}} \sum_{j\in\mathbb{Z}_{k/2},\, l\in(1/2)\mathbb{Z}_k} \tan\frac{(2j+1)\pi}{2(k+2)}\, S_j^l \chi_l(2\mathrm{i}t) \\ &= \sum_{j\in(1/2)\mathbb{Z}_k} (-1)^{2j}\chi_j(2\mathrm{i}t). \end{aligned} \tag{4.12}$$

The remarkable simplification of the last formula makes the interpretation of the CS inspired 2D model obvious. The Klein bottle diagram corresponds to projecting the SU(2) WZW model by a slight modification of the $\mathbb{Z}_2$ orbifold transformation used in the worldsheet orbifold interpretation of Ishibashi's model above. Namely, we now supplement the orbifold $\mathbb{Z}_2$ action leading to Ishibashi's model, by the action of the non-trivial central element of SU(2):

$$\Omega J_n^a \Omega^{-1} = \bar{J}_n^a, \qquad \Omega\,|j\otimes\bar{j}\rangle = (-1)^{2j}\,|\bar{j}\otimes j\rangle. \tag{4.13}$$

Hence, the 2D model that we have obtained from the SU(2) CS gauge theory can be interpreted as a worldsheet orbifold model, in which the simplest orbifold action of (4.1) is combined with the target action $g \mapsto -g$ on SU(2), a $\mathbb{Z}_2$ mapping known from the context of extended chiral algebras [19].

The one-loop amplitudes of any worldsheet orbifold model should satisfy the consistency conditions (2.17). Possibilities for given amplitudes to satisfy these constraints depend on the (Chan–Paton) degeneration of open sectors of the model. Thus, in the model that corresponds to (4.13) and (3.4), each of the two components of the singular locus is equivalent to $W_0(\ell) + W_{k/2}(\ell)$, and fusion of the two components of the singular locus produces both components of the open-string Hilbert space, $[\phi_0]$ and $[\phi_{k/2}]$, in duplicate. This degeneration of the open-string spectrum is the correct degeneration required in order to obey (2.17). We have thus confirmed that in this specific example, the correspondence between CS gauge theory and 2D worldsheet orbifolds does indeed allow one to identify the proper Chan–Paton degeneration of the open sector of the theory.

4.2. Extended chiral algebras and 3D orbifolds

We have now identified the CFT that corresponds to the SU(2) CS gauge theory on orbifolds. This CFT is actually a modification of the Ishibashi model of open-strings on the SU(2) group manifold. This makes us wonder whether Ishibashi's model itself can also be classified with the use of CS gauge theory on orbifolds.

The only difference between the two worldsheet orbifold models as defined by (4.1) and (4.13) is a $\mathbb{Z}_2$ twist,

$$|j\otimes\bar{j}\rangle \to (-1)^{2j}\,|j\otimes\bar{j}\rangle. \tag{4.14}$$

If taken as a $\mathbb{Z}_2$ orbifold action in the SU(2) WZW model on closed oriented surfaces, this $\mathbb{Z}_2$ twist turns the SU(2) model into an SO(3) WZW model. According to [19], the SO(3) WZW model corresponds to the SO(3) CS gauge theory on manifolds. This gives us actually a clue about the CS gauge theory for Ishibashi's model. To follow this clue, we consider the SU(2) CS gauge theory, but now we build in the $\mathbb{Z}_2$ twist difference between (4.1) and (4.13) by fusing each component of the singular locus with the Wilson line that carries the "algebra-extending" representation of spin $\frac{1}{4}k$.

To confirm that the CFT corresponding to this peculiar CS gauge theory is indeed the model discovered by Ishibashi, we will quantize the theory on $\mathcal{O}_C \times \mathbb{R}$. In accord with its definition, our theory now allows for just those holonomies h around singular points that square to minus one as elements of SU(2). Consequently, the singular locus can carry just one representation, of spin $\frac{1}{4}k$. We can infer the spectrum on the thickened open string from the SU(2) fusion rules (3.14) of the relevant representations carried by the two singular points of $\mathcal{O}_C$:

$$[\phi_{k/4}] \times [\phi_{k/4}] = \sum_{j \in \mathbb{Z}_{k/2}} [\phi_j], \tag{4.15}$$

in accord with the structure of Ishibashi's model.

5. More examples: Chern–Simons orbifold zoo

Moore and Seiberg conjectured an appealing classification [19] claiming that every CFT (or at least target orbifolds and cosets) can be incorporated into the CS approach to CFT. In the orbifold case, the relevant CS gauge theories are those with multiply connected gauge groups (for orbifolds leading to extensions of chiral algebra), and with disconnected gauge groups of the structure $G \otimes \mathcal{G}$, where the orbifold group G acts on $\mathcal{G}$ via automorphisms. It would be nice to have a similar classification for worldsheet orbifolds as well.

Before approaching this issue, it will be instructive to extend the class of examples discussed so far. Up to now, we have studied CS theory with non-abelian gauge groups that lead to WZW models; in this section, we will study $c = 1$ CFTs and orbifolds thereof. The structure of these models will clarify the question of how exotic worldsheet orbifolds can be described via CS gauge theory. This question is not only interesting in itself, but also provides some hints about the incorporation of worldsheet orbifolds into the classification given by Moore and Seiberg.

5.1. U(1) *Theory and* $c = 1$ *worldsheet orbifolds*

CFTs with $c = 1$ correspond to U(1), or rather to O(2), CS gauge theory [19]. The U(1) CS gauge theory is an example of the theory with a multiply connected gauge group, and thus corresponds to CFT with an extended chiral algebra, the chiral algebra of the rational torus [19]. Let us first recall some basic facts about the rational torus. This model corresponds to strings in one target dimension X, $X \equiv X + 2\pi R$. For any value of R, this model has the U(1) Kac–Moody symmetry. In our normalization and notation, the $\widehat{U(1)}$ primaries are

$$\phi_{m,n}(z, \bar{z}) = \exp\{p_L X_L(z)\} \exp\{p_R X_R(\bar{z})\}, \tag{5.1}$$

with left- and right-momenta

$$(p_L, p_R)_{\phi_{m,n}} = \left(\frac{m}{2R} + nR, \frac{m}{2R} - nR\right), \quad m, n \in \mathbb{Z}. \tag{5.2}$$

For rational values of $2R^2$, say p/q, $\phi_{p,q}$ be come chiral, and extend the chiral current algebra of U(1) to the chiral algebra of the rational torus generated by $\exp\{i\sqrt{2N} X_L(z)\}$ and $\partial X(z)$, with $N = pq$. The rational-torus CFT contains $2N$ primaries ϕ_r of this chiral algebra:

$$\phi_r = \exp\left\{\frac{ir}{\sqrt{2N}} X(z)\right\}, \quad r = 0, 1, \ldots, 2N. \tag{5.3}$$

Diagonal modular invariants correspond to p or q equal to one. (See [19] for details.)

Quantization of the U(1) CS gauge theory on $\mathcal{O}_C \times \mathbb{R}$ is quite analogous to the case of SU(2) theory we discussed above. At level N, the singular locus can carry any representation whose holonomy squares to one. We obtain two representations, namely ϕ_0 and ϕ_N. Fusing the representations carried by the two components of the singular locus according to fusion rules of the rational torus:

$$[\phi_r] \times [\phi_s] = [\phi_{r+s}], \quad r, s, r+s \in \mathbb{Z}_{2N}, \tag{5.4}$$

we get the Hilbert space of U(1) CS gauge theory on $\mathcal{O}_C$:

$$\mathcal{H}_{\mathcal{O}_C} = 2\{[\phi_0] \oplus [\phi_N]\}. \tag{5.5}$$

In accord with our discussion in the previous sections, it should be isomorphic to the spectrum of open states of a $c = 1$ worldsheet orbifold.

Worldsheet orbifolds of the (rational) torus were discussed in [4,7]. Motivated by the conjectured correspondence with CS gauge theory, we are mainly interested in $\mathbb{Z}_2$ orbifolds of models with diagonal modular invariants. There are essentially two such (classes of) models of importance to us, one standard and one exotic. The standard one uses the world-sheet parity group as the orbifold group, and the exotic one supplements the parity action by the target reflection,

$$X \mapsto -X. \tag{5.6}$$

(For details, see [4,7].) These two models are dual to each other, i.e. they are isomorphic up to redefinition $R \to 1/2R$ of the target radii. These two dual pictures of the same system can be used to shed some light on each other. In particular, we have a simple geometrical interpretation of the spectrum of open states of the model [7]: open strings are (half)-winding states with their ends sitting in either of the two fixed points of the orbifold involution (5.6). In particular, this simple picture elucidates the structure of the (Chan–Paton) degeneration of the open strings in the model, which is now related to the existence of two fixed points of

(5.6). As we are now going to see, these results can be reproduced from quantization of CS gauge theory on orbifolds, where the Chan–Paton degeneration comes from different ways in which one representation can be obtained by fusion of the Wilson lines that represent the string boundaries in CS gauge theory.

Let us first consider the "standard" $c = 1$ worldsheet orbifold at $R^2 = 1/2N$, which corresponds to a diagonal modular invariant of the orbifold model. The open spectrum of the model contains states with momenta

$$p_{\rm open} = \frac{m}{2R} \equiv \frac{m\sqrt{2N}}{2}. \tag{5.7}$$

The spectrum can be decomposed into two irreducible representations of the symmetry algebra of the orbifold model, depending on whether m is even or odd. These two representations are exactly the two representations of (5.5). Consequently, the U(1) CS gauge theory on orbifolds as discussed in (5.5) corresponds to the "standard" $c = 1$ worldsheet orbifold at radius $R = 1/\sqrt{2N}$. In particular, the factor of two in front of the right-hand side of (5.5) gives the Chan–Paton degeneracy and counts the fixed points of (5.6) in the dual picture of the model.

At $R^2 = \frac{1}{2}N$, the spectrum of open states of the standard worldsheet orbifold model carries momenta

$$p_{\rm open} = \frac{m}{2R} \equiv \frac{m}{\sqrt{2N}}, \tag{5.8}$$

and can be decomposed into $2N$ representations of the symmetry algebra of the model. To get a CS gauge theory description of this region of large target radii, we will proceed in analogy with the analysis of the SU(2) worldsheet orbifolds in the previous section. We know from the duality mentioned above that we are looking in fact for a CS interpetation of the exotic worldsheet orbifold at $R^2 = 1/2N$. Thus, we will construct a new U(1) CS gauge theory on orbifolds as follows. We supplement the $\mathbb{Z}_2$ action on manifolds by a $\mathbb{Z}_2$ twist, which is a CS analogy of the target transformation (5.6): Over the singular locus, we extend the orbifold group to $O(2) = \mathbb{Z}_2 \otimes U(1)$ and allow only those holonomies that do not belong to the U(1) subgroup in O(2); all other holonomies take values in U(1). This prescription defines a gauge theory, in which the singular points of $\mathcal{O}_C$ are now labeled by twisted primaries of the chiral algebra of the $\mathbb{Z}_2$ (target) orbifold [6]. Using the relevant fusion rules: [6]

$$\begin{aligned}
&[\sigma_i]\times[\sigma_i] = [1] + [\phi_N^i] + \sum_{r\ \rm even}[\phi_r], &\quad& [\sigma_1]\times[\sigma_2] = \sum_{r\ \rm odd}[\phi_r],\\
&[\tau_i]\times[\tau_i] = [1] + [\phi_N^i] + \sum_{r\ \rm even}[\phi_r], && [\tau_1]\times[\tau_2] = \sum_{r\ \rm odd}[\phi_r],\\
&[\sigma_i]\times[\tau_i] = [j] + [\phi_N^{i+1}] + \sum_{r\ \rm even}[\phi_r], && [\sigma_i]\times[\tau_{i+1}] = \sum_{r\ \rm odd}[\phi_r],
\end{aligned} \tag{5.9}$$

and composing the result into representations of the rational-torus chiral algebra, one gets exactly the spectrum (5.8) of the $2N$ representations of the standard worldsheet orbifold at

[6] Our notation here is that of [6].

$R^2 = \frac{1}{2}N$, with a Chan–Paton degeneration of the open sector. Thus, the exotic worldsheet orbifold at small radius, or alternatively the standard orbifold at large radius, corresponds to the twisted U(1) CS gauge theory we have just constructed.

Two interesting consistency checks can be made immediately. First, notice that using O(2) as the gauge group on 3D orbifolds, one can easily recover the model that corresponds to open strings on target orbifold $S^1/\mathbb{Z}_2$, which mixes in the obvious way the two $c = 1$ models just discussed, producing simultaneously the $\mathbb{Z}_2$-twisted closed states, as necessary. Secondly, notice that the compact boson at the self-dual radius, $R = \frac{1}{2}$, can be reconstructed from each of the two CS gauge theories presented above. It is reassuring that both descriptions give the same result. This closes our study of $c = 1$ worldsheet orbifolds via CS gauge theory.

5.2. *Discrete gauge groups*

Many crucial points of the relation between CFTs on surfaces with boundaries and/or crosscaps (i.e. worldsheet orbifold models) on one hand and the 3D CS gauge theory on the other can be efficiently isolated by studying holomorphic orbifolds [6]. At the level of CS gauge theory, holomorphic orbifolds are described by discrete gauge groups [18,19].

Let us consider the CS gauge theory with an arbitrary finite gauge group G, on $\mathbb{Z}_2$ orbifolds. I will limit the discussion to the classical theory; quantization can be treated similarly as in [18], after a choice of an element of $H^4_{\mathbb{Z}_2}(B(\mathbb{Z}_2, G), \mathbb{Z})$ is made, to represent the choice of a Lagrangian (see Appendix B).

The phase space for canonical quantization on an orbifold $\mathcal{O}$ is given by the space of flat principal G-bundles over $\mathcal{O}$. On the thickened open string $\mathcal{O}_C$, flat principal G-bundles are classified by representations of the fundamental group $\pi_1(\mathcal{O}_C)$ in the gauge group:

$$\mathbb{Z}_2 * \mathbb{Z}_2 \to G. \tag{5.10}$$

This recovers the picture of standard worldsheet orbifolds in two dimensions as discussed is Section 2.1: Flat G-bundles over $\mathcal{O}_C$ are in one-to-one correspondence with the monodromies (2.12) of the fields on the open string, i.e. they are in one-to-one correspondence with the open twisted states of a standard worldsheet orbifold (cf. Section 2).

Exotic worldsheet orbifolds can also be obtained in a simple way. To construct the CS gauge theory corresponding to a (holomorphic) exotic orbifold with orbifold group $G \subset \tilde{G} \times \mathbb{Z}_2^{\mathrm{ws}}$, we have to sum over a restricted class of $\tilde{G}$-bundles over $\mathbb{Z}_2$ orbifolds. This restriction corresponds to the commutativity restriction discussed in Section 2.1 (cf. (2.11)). Upon denoting by G_0 the set of all the elements from G that act trivially on the worldsheet and thus represent a target orbifold group, G can be written as $\mathbb{Z}_2 \otimes G_0$. Given now an orbifold $\mathcal{O}$, its fundamental group $\pi_1(\mathcal{O})$ has the structure of a $\mathbb{Z}_2$ extension of $\pi_1(\overline{\mathcal{O}})$, the fundamental group of its double cover $\overline{\mathcal{O}}$ (which is, by assumption, a manifold). The allowed holonomies are now required to respect these $\mathbb{Z}_2$ extensions on G and $\pi_1(\mathcal{O})$, i.e. they should make the following diagram commutative:

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(\overline{\mathcal{O}}) & \to & \pi_1(\mathcal{O}) & \to & \mathbb{Z}_2 & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} & & \\ 1 & \to & G_0 & \to & G & \to & \mathbb{Z}_2 & \to & 1 \end{array} \tag{5.11}$$

This commutative diagram thus defines a variant of CS gauge theory, in which the gauge group is intertwined non-trivially with the action of the orbifold group $\mathbb{Z}_2$. Hence, the exotic holomorphic worldsheet orbifolds can be given a three dimensional CS description, with the exact form of the allowed holonomies is encoded in the requirement of commutativity of (5.11). I will speculate on the nature of this exotic version of gauge theory in Section 5.3.

5.3. Gauging a mapping class group

In Sections 5.1 and 5.2 we have seen examples of worldsheet orbifold CFTs whose corresponding CS gauge theories apparently intertwine in a non-trivial way the gauge group with the action of the orbifold group $\mathbb{Z}_2$. For example, the CS gauge theory that was shown to correspond to (5.8) is neither an O(2) gauge theory, nor a U(1) theory on $\mathbb{Z}_2$ orbifolds, since the holonomies around singular points are treated differently from the holonomies around the non-contractible circles of the conventional manifold origin. I will now argue that the proper way how to interpret such a theory is to think of $\mathbb{Z}_2$ as a part of the gauge group, in a very specific sense that amounts to gauging the $\mathbb{Z}_2$ as a part of the mapping class group of the underlying manifold.

Before discussing the construction, however, I will present yet another heuristic argument that $\mathbb{Z}_2^{\text{ws}}$ should be treated as a part of the gauge group. It is shown in Appendix B that consistent Lagrangians for CS gauge theory on $\mathbb{Z}_2$ orbifolds are classified by elements of the fourth $\mathbb{Z}_2$ equivariant cohomology ring $H^4_{\mathbb{Z}_2}(B(\mathbb{Z}_2,\mathcal{G}),\mathbb{Z})$, where $\mathcal{G}$ is the original CS gauge group, and $B(\mathbb{Z}_2,\mathcal{G})$ is tom Dieck's classifying space of principal $\mathcal{G}$-bundles over $\mathbb{Z}_2$-manifolds. Recalling that

$$H^*_{\mathbb{Z}_2}(B(\mathbb{Z}_2,\mathcal{G}),\mathbb{Z}) = H^*(B(\mathbb{Z}_2\times\mathcal{G}),\mathbb{Z}) \tag{5.12}$$

(see Appendix B), it is easy to see that the right-hand side is exactly the object that classifies consistent Lagrangians for the gauge group $\mathbb{Z}_2\times\mathcal{G}$ on manifolds. We might interpret this fact as a signal that the $\mathbb{Z}_2$ group acting on manifolds has become a part of the gauge group.

We are free to define a gauge theory that corresponds to the ideas presented above, as follows. In the standard construction of a gauge theory whose gauge fields correspond to connections A on a principal $\mathcal{G}$-bundle, the functional integral that defines the theory on a given manifold M contains summation over all principal $\mathcal{G}$-bundles on M:

$$Z(M) = \sum_{\mathcal{G}-\text{bundles}} \int_{\mathcal{A}} \mathrm{D}A\,\mathrm{e}^{\mathrm{i}\mathcal{S}(A)}, \tag{5.13}$$

where $\mathcal{A}$ denotes the set of gauge equivalence classes of the connection, $\mathcal{S}(A)$ is a gauge invariant Lagrangian, and a gauge fixing procedure which gives sense to the formal functional integral is implicitly assumed. In paricular, for finite gauge groups [18,33] this summation distinguishes the gauge theory from the theory with the discrete symmetry being just global.

We can now define the theory with a mapping class group gauged, as a simple extension of the construction just discussed. Let us consider a $\mathbb{Z}_2$-extension of $\mathcal{G}$, $\mathbb{Z}_2 \otimes \mathcal{G}$. With this as a gauge group, the sum in (5.13) would run over all $\mathbb{Z}_2 \otimes \mathcal{G}$ principal bundles. Recalling that principal $\mathbb{Z}_2 \otimes \mathcal{G}$ bundles are spaces with free $\mathbb{Z}_2 \otimes \mathcal{G}$ actions, we will now modify the sum so as it will now run over all $\mathbb{Z}_2 \otimes \mathcal{G}$-spaces that are $\mathcal{G}$-free, but not necessarily $\mathbb{Z}_2 \otimes \mathcal{G}$-free. In other words, these spaces can be thought of as (total spaces of) principal $\mathcal{G}$-bundles with a G-action on them. Thus, instead of summing over principal bundles classified by the classifying space $B(G \times \mathcal{G})$, which would correspond to the conventional $\mathbb{Z}_2 \otimes \mathcal{G}$ gauge theory, we are summing in the "exotic" version of gauge theory over the objects classified by tom Dieck's classifying space $B(G, \mathcal{G})$. In this case, $\mathbb{Z}_2$ acts on 3D manifolds as an element of their mapping class group, which explains the title of Section 5.3. The CS gauge theory on $\mathbb{Z}_2$ orbifolds as discussed in the previous sections confirms that this approach really makes sense, since the theory represents a concrete example of the formal definition of the "exotic" gauge theory. In particular, the CS gauge theories discussed in Sections 5.1 and 5.2 are examples of gauge theories of this type. Hence, this extension of the standard definition of CS gauge theory allows us to add the CFTs of worldsheet orbifolds to Moore and Seiberg's list of 2D conformal field theories classifiable by their corresponding 3D CS gauge theories.

6. Concluding remarks

In this paper I have studied CS gauge theory on 3D $\mathbb{Z}_2$ orbifolds. This theory is interesting not only because it satisfies the axioms of equivariant topological field theory [11], but especially because it is intimately related to 2D CFT on surfaces with boundaries and/or crosscaps (the so-called "worldsheet orbifold" CFTs). This relation gives us new insight into several aspects of open-string theory; it may also have implications for the boundary conformal scattering in two dimensions, a problem that itself has many interesting ramifications [13].

We have seen that the 2D and 3D aspects of CS gauge theory on orbifolds illuminate each other in an interesting way:

For one, the 2D description of 2D CFT reveals the geometrical origin of the Chan–Paton mechanism (responsible for the existence of space–time Yang–Mills gauge symmetry in open-string theory). From this point of view, the rationale for the existence of the Chan–Paton symmetry is in 3D algebraic topology, namely in the existence of twisted principal $\mathcal{G}$-bundles of the corresponding CS gauge group $\mathcal{G}$ (not to be confused with the resulting Chan–Paton gauge group) on 3D $\mathbb{Z}_2$ orbifolds. Moreover, our interpretation of the open-string boundary as a link of Wilson lines in CS gauge theory leads to a surprisingly simple prescription for the identification of open string spectra in 2D CFTs, in terms of fusing specific Wilson lines in CS gauge theory.

On the other hand, the analysis of several specific 2D CFTs (related to the so-called "exotic worldsheet orbifolds") in fact suggests the existence of a more unified treatment for the 3D gauge theory, in which the original gauge group and the orbifold $\mathbb{Z}_2$ group become two parts of a larger gauge group. Conceptually, this step can be considered an extension of the standard definition of gauge theories. This extended class of 3D CS gauge theories then allows us to extend the classification results of [19] to open-string theory, proving in particular that at least those open-string models that can be interpreted as worldsheet orbifold models in the sense of Section 2 do fit into the general classification by Moore and Seiberg that uses 3D CS gauge theory to classify 2D CFTs.

Note also that in this paper, we have constructed a quantum field theory on spaces with singularities. This might be particularly interesting when combined with studies of 2 + 1-dimensional quantum gravity which can be formulated as CS gauge theory with a non-compact gauge group [23,28,32]. In fact, 2 + 1-dimensional quantum gravity on orbifolds (not necessarily $\mathbb{Z}_2$ ones) might give us an exactly soluble quantum theory with (mild) space–time singularities completely under control. In general, the effective equivalence between orbifold singularities and Wilson lines as seen in this paper may be considered a toy example of the idea that black holes (represented by the singular locus in this toy example) have just as much of hair as any particle [33], simply because the singularities are equivalent to a sum over Wilson lines that represent physical particles.

This paper is just a small step towards the complete CS gauge theory on orbifolds and its full correspondence with 2D CFT on surfaces with boundaries and/or crosscaps. The reader undoubtedly noticed that I have frequently chosen the way of smallest resistance instead of considering the most general situation possible. Indeed, the focus of this paper has been on the main line of arguments that leads as effectively as possible from CS gauge theory on orbifolds to CFTs of worldsheet orbifolds, and allows us to discuss specific examples. Many interesting aspects of the story had to be left out for future investigation.

Appendix A. Geometry of 3D orbifolds

The CS gauge theory is a theory of (flat) connections on 3D "space–times". To be able to study the theory on "space–times" that are orbifolds, we need some basic elements of orbifold geometry.

An n-dimensional orbifold is defined as a space modeled locally by factors of domains in $\mathbb{R}^n$ by discrete groups. More precisely, we will define an orbifold $\mathcal{O}$ as an underlying Hausdorff topological space $X_{\mathcal{O}}$ with a maximal atlas of coverings by open sets $\{U_i\}$. If $\mathcal{O}$ were a manifold, the U_is would be open subsets in $\mathbb{R}^n$. In the case of orbifolds, we associate with each U_i a discrete group G_i, such that U_i is a factor of a domain $\overline{U}_i \subset \mathbb{R}^n$ by G_i,

$$U_i = \overline{U}_i / G_i. \tag{A.1}$$

(To avoid some counter-intuitive cases, we require that G_i act on $\overline{U}_i$ effectively.) Maps between charts are required to respect the group action.

For each point x in an orbifold $X_{\mathcal{O}}$, the smallest group G_i associated to a domain containing x is called the "isotropy group" of x. The subset in $X_{\mathcal{O}}$ of points whose isotropy group is non-trivial is called the locus of singular points, or the "singular locus" of $\mathcal{O}$.

To be able to define fibered bundles over orbifolds, a structure that we need in gauge theory, we must first define the notion of morphisms between orbifolds. A "morphism" from orbifold $\mathcal{O}$ to another orbifold $\mathcal{O}'$ is defined as a mapping f between the underlying spaces, $f : X_{\mathcal{O}} \to X_{\mathcal{O}'}$, that respects the orbifold structure of $\mathcal{O}$ and $\mathcal{O}'$, i.e. it respects the group action in each coordinate chart. As a consequence, if x is an arbitrary point in $\mathcal{O}$ with isotropy group G_x and $y = f(x)$, then necessarily the isotropy group G_y of y contains G_x as a subgroup.

This definition of morphisms between orbifolds gives us a category of orbifolds, in which such notions as covering maps, fibered bundles, homotopies between maps, homotopy groups, etc. can be straightforwardly defined. For example, the mapping that retracts the thickened open string $\mathcal{O}_C$ to the open string $\mathcal{O}_S$ itself (see Fig. 2) represents a homotopy from $\mathcal{O}_C$ to $\mathcal{O}_S$ (in fact, this map is a deformation retraction; for the definition of the latter in the case of manifolds, see e.g. [34]). This fact explains the observation made in the paper that the orbifold fundamental groups of $\mathcal{O}_C$ and $\mathcal{O}_S$ are isomorphic.

The category of orbifolds is very similar to the category of G-spaces with G-equivariant maps as morphisms (here G is an a priori fixed finite group). For the purposes of this paper, these two categories can be considered in many respects equivalent.

Let us now proceed from the topology of orbifolds to their geometry. We define a principal $\mathcal{G}$-bundle for any (Lie) group $\mathcal{G}$ over an orbifold $\mathcal{O}$ as follows. Let P be an orbifold fibered over an orbifold $\mathcal{O}$ (i.e. the projection $\pi : \mathcal{P} \to \mathcal{O}$ is an orbifold morphism), such that for each chart U_i on $\mathcal{O}$ one is given a representation of G_i in $\mathcal{G}$, and for U_i from a sufficiently refined covering of $\mathcal{O}$, we have

$$\pi^{-1}(U_i) = (\overline{U}_i \times \mathcal{G})/G_i, \tag{A.2}$$

where the action of G_i on U_i is that of (A.1), and the action on $\mathcal{G}$ is given by the representation of G_i in $\mathcal{G}$. Then $\mathcal{P}$ is what we can call the total space of a principal $\mathcal{G}$-bundle over $\mathcal{O}$.

To be more specific, let us illustrate the definition of the principal bundle by classifying principal SU(2) bundles over our favorite orbifold $\mathcal{O}_C$. To construct an SU(2) principal bundle over $\mathcal{O}_C$, we have to specify a representation of the orbifold group $\mathbb{Z}_2$ in SU(2), over each singular point in $\mathcal{O}_C$. There are two such representations possible, over each singular point. One of them is trivial and maps $\mathbb{Z}_2$ to the identity in SU(2), and the other one maps $\mathbb{Z}_2$ to the center of SU(2). These two representations represent two possible twists of a principal SU(2) bundle over a singular point of $\mathcal{O}_C$. One of them gives a topologically trivial bundle over a vicinity of the singular point, while the other one is "twisted," and effectively reduces the structure group over the singular point from SU(2) to SO(3). Note the amusing fact that in the twisted case, the total space of the principal bundle over a vicinity of the singular point is a manifold, and the singularity of the bundle is due to a singular projection to the base orbifold $\mathcal{O}_C$.

Appendix B. Lagrangians of CS gauge theory on orbifolds

In this appendix I present some technicalities of the definition of CS Lagrangians for general (compact) gauge group, not necessarily connected or simply connected, on 3D $\mathbb{Z}_2$ orbifolds. The analysis follows closely the non-equivariant case discussed in [18].

Regardless of what the gauge group is, the requirements of factorization in the theory on orbifolds make the Lagrangian $S(A)$ on $\mathcal{O}$ zero if $\mathcal{O}$ is the boundary of a 4D $\mathbb{Z}_2$ orbifold $\mathcal{B}$ assuming A can be extended as a flat connection over $\mathcal{B}$. First we will check whether there exists an obstruction for a 3D $\mathbb{Z}_2$ orbifold to be the boundary of a 4D $\mathbb{Z}_2$ orbifold. If such an obstruction existed, it would be an element of the third $\mathbb{Z}_2$ equivariant cobordism group $I_3(\mathbb{Z}_2)$ (see [35]). In general, $I_*(\mathbb{Z}_2)$ is defined as the group of equivalence classes of $\mathbb{Z}_2$ manifolds, two of them being equivalent if they bound a $\mathbb{Z}_2$ manifold. Using a split exact sequence [35], the cobordism group of our interest can be easily calculated, leading to $I_3(\mathbb{Z}_2) = \mathbb{Z}_2$. Hence, there is a $\mathbb{Z}_2$ obstruction for some 3D $\mathbb{Z}_2$ orbifolds to represent the boundary of a 4D $\mathbb{Z}_2$ orbifold, which indicates that the definition of Lagrangian must be treated carefully. To this goal, we will modify to the orbifold case the results of [18], where the Lagrangian for CS gauge theory on manifolds has been defined using group cohomology.

Let us consider CS gauge theory on $\mathbb{Z}_2$ orbifolds with a compact gauge group $\mathcal{G}$, not necessarily connected or simply connected. In the case of the theory on manifolds, the principal bundles over a given manifold M are classified by homotopy classes of mapping of M to the "classifying space" $B\mathcal{G}$, and the consistent Lagrangians are classified by the fourth cohomology group $H^4(B\mathcal{G}, \mathbb{Z})$ [18].

For principal bundles over orbifolds, relevant classifying spaces were defined and studied by tom Dieck in [31] (see also [36]). His classifying space $B(\mathbb{Z}_2, \mathcal{G})$ has the property that for any principal $\mathcal{G}$-bundle E over a manifold B with a $\mathbb{Z}_2$ action on E and B commuting with the $\mathcal{G}$-action on E, there exists a $\mathbb{Z}_2$ equivariant mapping (unique up to $\mathbb{Z}_2$ equivariant homotopy) of B to $B(\mathbb{Z}_2, \mathcal{G})$, which induces E on B from a universal bundle over $B(\mathbb{Z}_2, \mathcal{G})$.[7] I now claim that the consistent Lagrangians of the CS gauge theory on orbifolds are classified by elements of the fourth equivariant cohomology [37] of tom Dieck's classifying space, $H^4_{\mathbb{Z}_2}(B(\mathbb{Z}_2, \mathcal{G}), \mathbb{Z})$.

To show this, let us first compute the relevant cohomology group. As shown in [31], the classifying space $B(G, \mathcal{G})$ is homotopic to the classifying space $B\mathcal{G}$, once the action of G on $B(G, \mathcal{G})$ is ignored. Thus, we can find a representant of $B\mathcal{G}$ such that G acts on it, and

$$H^4_{\mathbb{Z}_2}(B(\mathbb{Z}_2, \mathcal{G}), \mathbb{Z}) = H^4_{\mathbb{Z}_2}(B\mathcal{G}, \mathbb{Z}). \tag{B.1}$$

With the use of the definition of equivariant cohomologies, we easily obtain

$$\begin{aligned} H^*_{\mathbb{Z}_2}(B(\mathbb{Z}_2, \mathcal{G}), \mathbb{Z}) &= H^*((B\mathcal{G} \times E\mathbb{Z}_2)/\mathbb{Z}_2, \mathbb{Z}) = H^*(((E\mathcal{G}/\mathcal{G}) \times E\mathbb{Z}_2)/\mathbb{Z}_2, \mathbb{Z}) \\ &= H^*((E\mathcal{G} \times E\mathbb{Z}_2)/(\mathbb{Z}_2 \times \mathcal{G}), \mathbb{Z}) = H^*(B(\mathbb{Z}_2 \times \mathcal{G}), \mathbb{Z}). \end{aligned} \tag{B.2}$$

Using now the Künneth formula [38] for integral cohomologies, we get the following result for the fourth cohomology group of our interest:

$$H^4_{\mathbb{Z}_2}(B\mathcal{G}, \mathbb{Z}) = H^4(B\mathcal{G}, \mathbb{Z}) \oplus H^2(B\mathcal{G}, \mathbb{Z}_2) \oplus \mathbb{Z}_2. \tag{B.3}$$

[7] Ref. [31] discusses a generalization of this construction to the case of general semi-direct products $G \otimes_\alpha \mathcal{G}$ as well, i.e. to principal $\mathcal{G}$-bundles with a G-action commuting with the $\mathcal{G}$-action on the total space up to a representation α of G in the group of $\mathcal{G}$-automorphisms. The corresponding classifying spaces, denoted as $B(G, \alpha, \mathcal{G})$, are relevant to the ideas of Section 5.3 of the paper.

Let us now consider a principal bundle E over an orbifold $\mathcal{O}$. Its double covering $\overline{E}$ over $\mathcal{O}$ is an example of the objects classified by $B(\mathbb{Z}_2, \mathcal{G})$. Let B is a four-manifold with the boundary $\overline{\mathcal{O}}$. Given a classifying map $\overline{\mathcal{O}} \to B(\mathbb{Z}_2, \mathcal{G})$ of E, there is an obstruction to extending it to an equivariant mapping $B \to B(\mathbb{Z}_2, \mathcal{G})$, given by the element $\gamma_*[\overline{\mathcal{O}}]$ of the third $\mathbb{Z}_2$ equivariant homology group $H_3^{\mathbb{Z}_2}(B(\mathbb{Z}_2, \mathcal{G}), \mathbb{Z})$. The torsion part of this group is isomorphic by the universal coefficients theorem to the torsion of $H^4_{\mathbb{Z}_2}(B(\mathbb{Z}_2, \mathcal{G}), \mathbb{Z})$ (see (B.3). Supposing for simplicity that the torsion of $H^4(B\mathcal{G}, \mathbb{Z})$ is $\mathbb{Z}_n$, then $2n \cdot \gamma_*[\overline{\mathcal{O}}]$ is homologically trivial in $B(\mathbb{Z}_2, \mathcal{G})$. We can thus define the Lagrangian on $\mathcal{O}$ by

$$2n \cdot S(A) = \frac{k}{8\pi} \int_P \mathrm{Tr}(F \wedge F), \tag{B.4}$$

where ∂P consists of $2n$ copies of $\overline{\mathcal{O}}$. A resolution of the $2n$-fold ambiguity of this definition is then given, recalling that the form $\Omega \equiv (k/8\pi^2)\mathrm{Tr}(F \wedge F)$ is in the image of the natural map $H^4_{\mathbb{Z}_2}(B(\mathbb{Z}_2, \mathcal{G}), \mathbb{Z}) \to H^4(B\mathcal{G}, \mathbb{R})$, by any element of the fourth equivariant cohomology of $B(\mathbb{Z}_2, \mathcal{G})$ as claimed above.

To summarize, the consistent Lagrangians for CS gauge theory with gauge group $\mathcal{G}$ on $\mathbb{Z}_2$ orbifolds are classified by the fourth $\mathbb{Z}_2$ equivariant integer cohomology group of the classifying space $B(\mathbb{Z}_2, \mathcal{G})$. For the purposes of the present paper (cf. Section 5.3), the most relevant point is that this cohomology group is isomorphic to the fourth integer cohomology group of the classifying space of $\mathbb{Z}_2 \times \mathcal{G}$.

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